The Shilnikov Saddle-Node Bifurcation in a Monetary Policy with Endogenous Time Preference

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Abstract

We improve the analysis made in Chang et al (2011), by exploring the possibilities for the raise of global indeterminacy via a Shilnikov saddle-node bifurcation on an invariant circle. This allows us to better understand the determinants for the emergence of endogenous fluctuations in a monetary policy model, and to explain the existence of irregular patterns. Hence, the economy may start at some point to oscillate around the long run equilibrium, and eventually deviate from its saddle-path stable solution, thus locating the economy in a particular optimal converging path that could not coincide with the one corresponding to the lowest desired interest rate.

1. Introduction

A bulk of literature is concerned with the capacity of monetary policy to stabilize the economy by ensuring the uniqueness of equilibrium. In the last two decades, paramount studies have shown that when the monetary authority decides to raise the nominal interest rate less than the increase in inflation (commonly known as a passive interest rate feedback rule), this might destabilize the economy by inducing fluctuations, and thus render the equilibrium indeterminate (see, Benhabib et al, 2001; Dupor, 2001). Most of these analyses simply consider a constant rate of time preference.

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One exception in found in Chang et al. (2011) who extend the Benhabib et al. (2001) model by exploring the role played by endogenous time preference in destabilizing the economy under monetary policy actions. Specifically, they find that either an active or a passive interest rate feedback rule can generate local indeterminacy of the equilibrium, which is in sharp contrast with the existing literature on the field.

More interestingly, recent literature is exploring the possibilities for the raise of *global* indeterminacy, where the economy may start at some point to follow completely different equilibrium paths towards the long-run steady state. The main implication for any policy decision is that if global indeterminacy occurs, public intervention becomes not sufficient to drive the economy towards a good long-run equilibrium. The agents' decisions, despite the initial conditions or other economic fundamentals, will locate the economy in a particular optimal converging path that could not coincide with the one corresponding to the lowest desired monetary policy. To date, only very few attempts have been made to analyze the conditions under which these indeterminacy problems arise outside the small neighborhood of the steady state (Mattana et al. 1999). The latter seems an innovative field to work on, even though it is usually related to very complicated nonlinear functions which increase the mathematical difficulties in handling these models.

The novelty of this paper relies in a deep investigation of the global behavior of the economy studied in Chang et al. (2011). To tackle this problem, we use the principles of bifurcation theory to gain hints on the global properties of the equilibrium, and to investigate the whole set of conditions which lead to the emergence of a quasi-periodic dynamics along a Shilnikov saddle-node bifurcation on an invariant circle. This bifurcation represents the route to creation and destruction of a periodic orbit in a smooth continuous-time dynamical system, and the transition from a resting behavior to bursting, i.e. a solution in which the variables undergo rapid oscillations followed by a period of quiescence, with both oscillation and quiescence continually repeated. In this kind of excitable system, a small perturbation above a certain threshold forces the equilibrium trajectories to make a large excursion before returning to rest. In this particular case, there is also a stable invariant circle which is the closure of the unstable manifold leaving the saddle point. This allows us to better understand the determinants for the emergence of endogenous fluctuations, and the existence of irregular patterns due to a sensitive dependence of our economy on the initial conditions.

The paper develops as follows. The second Section introduces the dynamic system associated with Chang et al. (2011). In the third Section, we prove the main proposition relating to the global properties of the equilibrium, and show the parametric onset for the emergence of global indeterminacy and the saddle-node dynamics. A brief conclusion reassesses the main findings of the paper, and a subsequent Appendix provides all the necessary proofs.

2. The Model

Assume that the economy consists of an infinitely lived representative household and a government, as in Chang et al. (2011). Time is continuous. Each household is endowed with one unit of labor (which is supplied inelastically), and derives utility from consumption, c, and real money holdings, m.

To begin with, let the utility function be defined as

$$U = \int_0^\infty u(c,m)e^{-\Delta}dt \quad (1)$$

Where

$$\Delta = \int_0^t \rho\big(u(c,m)\big)ds \quad (2)$$

is the cumulated subjective discount rate, depending on the instantaneous subjective discount rate, $\rho(u)$, at time s. Hence

$$\dot{\Delta} = \rho(u) \quad (3)$$

where the dot stands for time derivative.

The government budget constraint is finally given by

$$\dot{a} = y + (R-\pi)a - Rm - c - \tau \quad (4)$$

where a represents the total real wealth, R is the nominal interest rate, π is the inflation rate, and τ measures exogenous lump-sum taxes.

The household's intertemporal optimization problem, follows consequently

$$\max \int_0^\infty u(c,m)e^{-\Delta}dt$$
$$s.t.$$
$$\dot{a} = y + (R-\pi)a - Rm - c - \tau$$
$$\dot{\Delta} = \rho(u)$$

which implies the following Hamiltonian function

$$H = u(c,m) + \lambda[y + (R-\pi)a - Rm - c - \tau] + \varphi\rho(u) \quad (5)$$

where λ and φ are the costate variables.

Assuming an endowment of physical capital, k, production occurs according to the following functional relationship

$$y = f(k) \quad (6)$$

which is assumed strictly increasing and concave in k, i.e., $f_k > 0$ and $f_{kk} < 0$ Solution to the optimization problem allows too btain the following third-order autonomous system of differential equations

$$\dot{\lambda} = \lambda(\rho - f_k(k))$$
$$\dot{\varphi} = \varphi\rho - u(c,m) \qquad (S)$$
$$\dot{k} = f(k) - c(\lambda, \varphi, k)$$

Linearization of (S) around the equilibrium, and straightforward computations leads to

$$\mathbf{Tr(J)} = \rho + f_k - c_k \quad (8)$$
$$\mathbf{Det(J)} = \frac{\rho\lambda f_{kk}(u_{mm} - \rho' u_c u_m)}{\Omega(\psi' - 1)}[\psi'(1-\varepsilon) - (\theta - \varepsilon)] \quad (9)$$
$$\mathbf{B(J)} = \lambda[\rho\rho'(u_c c_\lambda + u_m m_\lambda - c_\lambda f_{kk})] + \rho(f_k - c_k) \quad (10)$$

where **Tr(J)**, **Det(J)** and **B(J)** are the trace, the determinant, and the sum of principal minors of order two, respectively.

Remark 1 Chang et al. (2011) points out that: (i) the equilibrium is determinate under an active interest rate feedback rule $(\psi' > 1)$; (ii) the equilibrium is determinate if the monetary authority implements an aggressively passive interest-rate feedback rule $(\psi' < \frac{\theta - \varepsilon}{1 - \varepsilon})$, while it may be either indeterminate or unstable if the monetary authority implements a moderately passive interest-rate feedback rule $(\frac{\theta - \varepsilon}{1 - \varepsilon} < \psi' < 1)$.

Specifically, an active interest-rate feedback rule $(\psi' > 1)$ implies that the inflation rate must fall. Hence households increase their money holding, whereas the time preference rate rises as well. As a consequence, the equilibrium solution is locally determinate. Alternatively, if monetary policy is passive $(\psi' < 1)$, a lower real interest rate is associated with an increase in inflation rate. This drives households to decrease their money holding, so that the time preference rate will fall. Finally, when the monetary authority implements a moderately passive interest-rate feedback rule $(\frac{(\theta - \varepsilon)}{(1 - \varepsilon)} < \psi' < 1)$, the effect of a reduction in the time preference rate will dominate the fall in the real interest rate, and consequently an indeterminate solution occurs.

3. The Shilnikov Saddle-Node Bifurcation

This section is devoted to analyze the Government policy actions devoted to drive an economy, originating arbitrarily close to the bad steady state, towards the virtuous steady state. This is a quite tough investigation to undertake, since the local analysis is not able to determine what happens to equilibrium trajectories outside a small neighborhood around the steady state. Our aim is therefore to detect the necessary conditions for which the monetary policy actions described in Chang et al. (2011) may induce a large amplitude limit cycle to coexist with a saddle point. In this case, a global bifurcation of orbits homoclinic to nonhyperbolic equilibria occurs, and complex dynamics do eventually emerge along with a saddle-node bifurcation on an invariant circle (i.e., the Shilnikov saddle-node bifurcation).

The bifurcation analysis provides several instruments to this end. To ease the mathematical computation, we can transform system (S) into a more convenient Jordan normal form in cylindrical coordinates (r, z, θ):

$$\dot{r} = a_1 rz + a_2 r^3 + a_3 rz^2$$
$$\dot{z} = b_1 r^2 + b_2 z^2 + b_3 z^3 + b_4 r^2 z \quad (11)$$
$$\dot{\theta} = \omega + c_1 z + c_2 r^2 + c_3 z^2$$

whose three-dimensional dynamics is topologically equivalent to the evolution of the original vector field in S (see, Wiggins, 1991).

In particular, r describes the amplitude of the limit cycle oscillations in the vicinity of the Hopf bifurcation. Noticeably, the first two equations are independent of θ, which describes a rotation around the r-axis with almost constant angular velocity $\dot{\theta} \approx \omega$, for any $|r|$ small. Thus, we can restrain the analysis to a simpler two-dimensional vector field, which is often called a truncated amplitude system:

$$\dot{r} = \hat{a} rz$$
$$\dot{z} = \hat{b} r^2 - z^2 \quad (12)$$

where $\hat{a} = -\frac{a_1}{b_2}$ and $\hat{b} = \pm 1$ (see [25]).

A Versal deformation of the normal form in (12) can be found, and the bifurcation phenomenon can be studied in the neighborhood of the origin. This is not, in general, a trivial task. For our system we can show the following

Proposition 1 *The transverse family*

$$\dot{r} = \mu_1 r + \hat{a} rz$$
$$\dot{z} = \mu_2 + \hat{b} r^2 - z^2 \quad (13)$$

is a versal deformation of system (12), and is topologically equivalent to the original system, S. Therefore, a non trivial equilibrium, E^*, occurs at the pitchfork curve

$$H = \left\{ (\mu_1, \mu_2) : \sqrt{\mu_2} = -\frac{\mu_1}{\hat{a}^2} \right\}$$

along which a saddle-node bifurcation emerges at $\mu_2 = 0$*, giving rise to two branches of fixed points* $\left(0, \pm\sqrt{\mu_2}\right)$.

Proof. See Wiggins (1991).

Interestingly, if the initial condition on capital is chosen in such a way that system *S* gives rise to a saddle-node bifurcation, then a continuum of equilibria can depart from a given initial condition of the predetermined variable, as clearly shown in Fig. 1, using the set of parameters in Chang et al. (2011).

Since this continuum of equilibria exists beyond the region relevant for the linear approximation of the dynamics in the neighborhood of the steady state, the result implies indeterminacy of global nature. Besides the result of global indeterminacy, the possibility that the model can exhibit this motion is of great interest also because the decomposition of the dynamics into phase/amplitude equations allows us to better understand the nature of the cyclical behavior of an economy where the effects of monetary policies on the long run properties of the equilibrium become totally unpredictable.
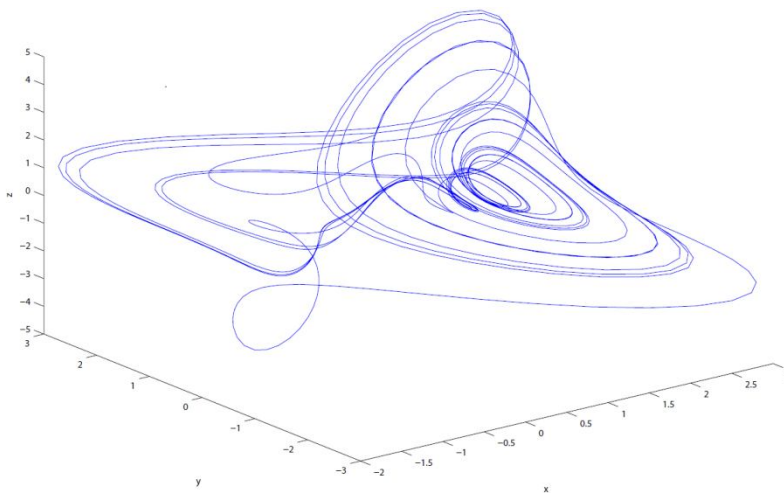

Fig. 1: The Shilnikov saddle-node bifurcation

4. Concluding Remarks

In this paper we have extended the Chang et al. (2011) monetary policy model with an endogenous rate of time preference. We found that the equilibrium can be indeterminate under an active interest-rate feedback rule and the equilibrium is determinate (indeterminate) if the monetary authority implements an aggressively (moderately) passive interest-rate feedback rule. These results provide insightful policy implications by exploring the possibilities for the raise of global indeterminacy via a Shilnikov saddle-node bifurcation on an invariant circle. In this case, the economy may start at some point to oscillate around the long run equilibrium, and eventually deviate from its saddle-path stable solution, thus locating the economy in a particular optimal converging path that could not coincide with the one corresponding to the lowest desired interest rate.

5. Appendix

Consider a second order Taylor expansion of the vector field in (S):

$$\begin{pmatrix} \dot{\tilde{\lambda}} \\ \dot{\tilde{\varphi}} \\ \dot{\tilde{k}} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \tilde{\lambda} \\ \tilde{\varphi} \\ \tilde{k} \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(\tilde{\lambda}, \tilde{\varphi}, \tilde{k}) \\ 0 \\ \tilde{f}_3(\tilde{\lambda}, \tilde{\varphi}, \tilde{k}) \end{pmatrix} \quad \text{(A.1)}$$

where $\tilde{f}_1(\tilde{\lambda}, \tilde{\varphi}, \tilde{k}) = -\frac{1}{2}\lambda\alpha(\alpha-1)(\alpha-2)k^{\alpha-3}\tilde{k}^2$ and $\tilde{f}_3(\tilde{\lambda}, \tilde{\varphi}, \tilde{k}) = \alpha(\alpha-1)k^{\alpha-2}\tilde{k}^2$.

Assume now that system (S) undergoes a triple-zero eigenvalue structure, which allows us to make the following change of coordinates

$$\begin{pmatrix} \tilde{\lambda} \\ \tilde{\varphi} \\ \tilde{k} \end{pmatrix} = \mathbf{T} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad \text{(A.2)}$$

via an appropriate transformation matrix

$$\mathbf{T} = \begin{bmatrix} u_1 & v_1 & z_1 \\ u_2 & v_2 & 0 \\ 1 & 0 & z_3 \end{bmatrix} \quad \text{(A.3)}$$

whose columns represent the eigenvectors associated to the triple-zero eigenvalues (see, Wiggins, 1991).

We are thus able to put (A.2) in a Jordan normal form

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} \bar{F}_1\left(w_1, w_2, w_{3,}\right) \\ \bar{F}_2\left(w_1, w_2, w_{3,}\right) \\ \bar{F}_3\left(w_1, w_2, w_{3,}\right) \end{pmatrix} \quad \text{(A.4)}$$

where:

$$\bar{F}_i\left(w_1, w_2, w_{3,}\right) = \frac{1}{D} \begin{pmatrix} \left(-v_2 z_3 A_1 + v_2 z_1 A_2\right)(w_1 + z_3 w_3)^2 \\ \left(u_2 z_3 A_1 - u_2 z_1 A_2\right)(w_1 + z_3 w_3)^2 \\ \left[v_2 A_1 + (-u_1 v_2 + u_2 v_1)A_2\right](w_1 + z_3 w_3)^2 \end{pmatrix} \quad \text{(A.5)}$$

with $D = v_2 z_1 - u_1 v_2 z_3 + u_2 v_1 z_3$, and $A_1 = -\frac{1}{2}\lambda\alpha(\alpha-1)(\alpha-2)k^{\alpha-3}$ and $A_2 = \alpha(\alpha-1)k^{\alpha-2}$.

Let us repeat the same procedure of above, and introduce a second transformation matrix

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & \delta \\ -\omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix} \quad \text{(A.6)}$$

which allows us to put system (A.4) into the normal form suitable to describe the presence of one zero and a pair of pure imaginary eigenvalues

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{pmatrix} \quad \text{(A.7)}$$

where $\tilde{F}_1 = B_1(w_1^2 + 2z_3 w_1 w_3 + z_3^2 w_3^2)$, $\tilde{F}_2 = B_2(w_1^2 + 2z_3 w_1 w_3 + z_3^2 w_3^2)$, $\tilde{F}_3 = B_3(w_1^2 + 2z_3 w_1 w_3 + z_3^2 w_3^2)$; and $B_1 = \frac{1}{D}(-v_2 z_3 A_1 + v_2 z_1 A_2)$, $B_2 = \frac{1}{D}(u_2 z_3 A_1 - u_2 z_1 A_2)$, $B_3 = \frac{1}{D}[v_2 A_1 + (-u_1 v_2 + u_2 v_1)A_2]$.

System (A.7) can be easily transformed into cylindrical coordinates

$$\dot{r} = a_1 rz + a_2 r^3 + a_3 rz^2$$
$$\dot{z} = b_1 r^2 + b_2 z^2 + b_3 z^3 + b_4 r^2 z \quad \text{(A.8)}$$
$$\dot{\theta} = \omega + c_1 z + c_2 r^2 + c_3 z^2$$

given $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$ (see Wiggins, 1991).

Additionally, the truncated-amplitude rescaled normal form can be derived from (A.8), keeping $\theta \,\square\, \omega$:

$$\dot{r} = \hat{a} rz$$
$$\dot{z} = \hat{b} r^2 - z^2 \quad \text{(A.9)}$$

where $\hat{a} = -\frac{a_1}{b_2}$ and $\hat{b} = \pm 1$.

A candidate for versal deformation of (A.9) is then

$$\dot{r} = \mu_1 r + \hat{a} rz$$
$$\dot{z} = \mu_2 + \hat{b} r^2 - z^2 \quad \text{(A.10)}$$

References

Benhabib, J., S. Schmitt-Grohé and M. Uribe (2001) Monetary Policy and Multiple Equilibria American Economic Review 91, 167-186.

Chang, W.Y., H.F. Tsai and J.J. Chang (2011) Endogenous time preference, interest-rate rules, and indeterminacy Japanese Economic Review 62, 348-364.

Dupor, B. (2001) Investment and Interest Rate Policy Journal of Economic Theory 98, 85-113.

Mattana, P., K. Nishimura and T. Shigoka (2009) Homoclinic bifurcation and global indeterminacy of equilibrium in a two-sector endogenous growth model International Journal of Economic Theory 5, 1-23.

Wiggins, S. (1991) Introduction to applied nonlinear dynamical systems and chaos, Springer-Verlag: New York.